Threeconnected graphs with only one Hamiltonian circuit[1]

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We will call graph *1-H-graph*, if it is threeconnected and it has only one Hamiltonian circuit *(H-circuit)*. We will say, that in the graph *G* three distinct vertices *x, y, z* in the given order comprise *special triplet* – shorter, *s-triplet {x, y, z}*, if

1) there is only one Hamiltonian chain *(H-chain) [x...y]* with end vertices *x, y*;
2) there is not *H*-chain *[x...z]*;
3) there either
   3.1) *G* is threeconnected; or
   3.2) *G* is not threeconnected, but it becomes threeconnected if vertex *t* and edges *tx*, *ty*, *tz* are added.

H-chains *[y...z]* can be of arbitrary number or be not at all.

Graph *G* satisfying these conditions will be called *preparation*.

If graphs *G* and *G'* without common elements have correspondingly *s*-triplets *{x, y, z}* and *{x',y', z'}*, then the *linking of the graphs G and G' through these triplets* will be called graph *G''*, that is built from graphs *G* and *G'*, that are joined with edges *xy'*, *yx'*, *zz'*.

G'' is *1-H*-graph. Because of condition 3 *G''* is threeconnected. The only *H*-circuit of *G''* is composed from *[x...y], yx', [x'...y'], y'x*.

Indeed, each *H*-circuit of *G''* has just two edges from *xy'*, *yx'*, *zz'*. Because of the condition 1 first two edges go only into indicated *H*-circuit. Because of the fact that there are not *H*-chains *[x...z]* in *G* and *[x'...z']* in *G'*, pairs of edges *xy'*, *zz'* and *yx'*, *zz'* do not go in any *H*-circuit of *G''*.

[1] This article is compiled from several fragments from Grinbergs manuscripts by D. Zeps

If *G* is graph with only one *H*-circuit we will say that the edges of the *H*-circuit are *strong*, but other edges are *weak*. For each vertex x of *G* with degree $p \geq 3$ there are at least *2(p-2)* triplets *x, y, z,* that satisfy condition 1 and 2 (Fig. 1, where strong edges are bold).

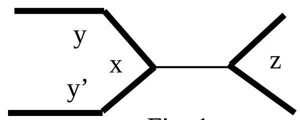
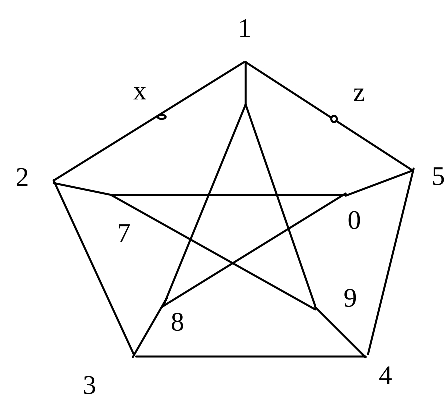


Fig. 1

y and *z* are taken correspondingly the end vertices of strong and weak edges *xy* and *xz*.
If preparation *G* have vertices of degree *2* then because of the condition 3.2 they all must go into *s*-triplet. But, if *G* is *1-H*-graph, the condition 3 is satisfied and each triplet of the type of fig. 1 is s-triplet; but there can be other *s*-triplets too. Two such graphs can be linked together in different ways and thus giving new *1-H*-graphs.
Thus, it is possible to build *1-H*-graphs with arbitrary large number of vertices.
Simplest graphs that we succeeded to find was some modifications of Petersen's graphs: G_0 with $n=9$, G_1 with $n=11$ and G_2, G_3 with $n=12$.* [The matrixes below which in *(i, j)* shows the number of *H*-chains between vertices *i* and *j* are computer data and added by us and in Grinbergs manuscripts naturally were absent. These matrixes allow easy to see that Grinberg characterized all s-triples in considered preparationes.]

1	2	3	4	5	6	7	8	9	0	x	y
0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	0	2	0	0	0	1	0	2
0	0	0	0	1	3	3	0	1	2	1	5
0	1	0	0	0	0	3	2	1	2	5	1
0	0	1	0	0	2	1	0	0	0	2	0
0	2	3	3	2	0	3	0	0	3	2	2
0	0	2	2	1	3	0	1	0	0	1	5
0	0	0	1	0	0	1	0	0	0	4	4
0	0	1	0	0	0	0	0	0	1	4	4
0	1	2	2	0	3	0	0	1	0	5	1
0	0	1	5	2	2	1	4	4	5	0	0
0	2	5	1	0	2	5	4	4	1	0	0

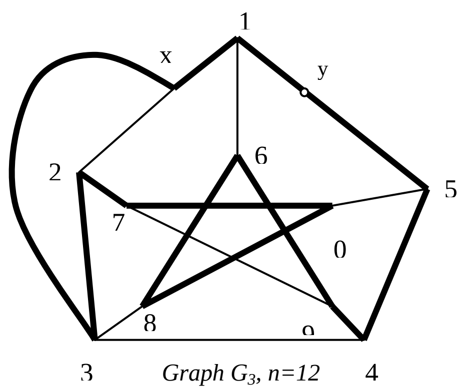


Graph G_2, n=12

Fig. 2

Here (in fig. 2) is *s*-triple *{x, 3, z}* (which with automorphisms of G_2 transforms into equivalent *s*-triples *{x, 7, z}*, *{*z*, 4, x}*, *{*z*, 0, x}*). Indeed, there are not *H*-chains [x...z] in other case there were *H*-circuits in the Petersen's graph. If we add edge *x3*, we get graph isomorphic to G_3 (in Fig. 3). In Fig.3 there is drawn the only *H*-circuit of the graph G_3, which has in corresponds the only *H*-chain of G_2 , namely, *[x...3]*.

1	2	3	4	5	6	7	8	9	0	x	y
0	0	1	2	0	0	1	1	1	2	1	1
0	0	1	5	2	6	1	3	1	2	0	7
1	1	0	0	1	3	2	0	2	3	1	6
2	5	0	0	1	5	4	3	1	5	5	4
0	2	1	1	0	2	1	1	1	0	2	1
0	6	3	5	2	0	3	1	1	4	2	3
1	1	2	4	1	3	0	2	0	1	1	6
1	3	0	3	1	1	2	0	2	1	4	8
1	1	2	1	1	1	0	2	0	3	5	6
2	2	3	5	0	4	1	1	3	0	6	3
1	0	1	5	2	2	1	4	5	6	0	1
1	7	6	4	1	3	6	8	6	3	1	0

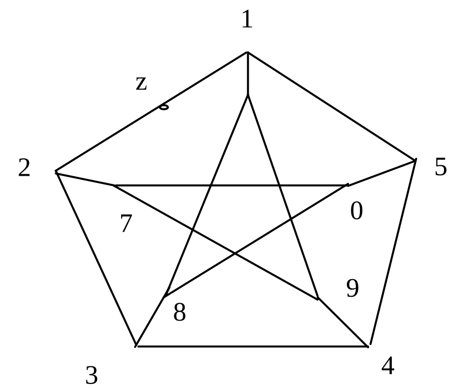


Graph G_3, n=12
Fig. 3

In the graph G_3 because of the condition 3.2 vertex *y* goes into each *s*-triple. From *y* goes out *H*-chain with ends in each other vertex of G_3 but only in vertices *1, 5* or *x* exactly one each case. Thus, one of these vertices can be first vertex of *s*-triple, but *y* must be the second in any case. Such are both trivial *s*-triples *{1, y, 6}* and *{5, y, 0}*. It can be established that there are two more *s*-triples *{1, y, 2}* and *{x, y, 2}* - making four *s*-triples together. Triples *{1, y, 5}* and *{5, y, 1}* are not *s*-triples because of condition 3.2. Because G_3 has only identical automorphism these *s*-triples are essentially different.

More simple preparation (G_1, fig. 4) with *s*-triple *{1, 4,z}*. Equivalent with vertex *4* are *8, 9* and *0*, because automorphisms with *(1)(2)(z)* are two: *(37)(40)(5)(6)(8 9)* and *(3)(7)(5 6)(4 8)(9 0)*.

1	2	3	4	5	6	7	8	9	0	z
0	0	2	1	0	0	2	1	1	1	0
0	0	0	1	2	2	0	1	1	1	0
2	0	0	0	4	4	4	0	3	3	2
1	1	0	0	0	3	3	2	0	2	6
0	2	4	0	0	4	4	3	3	0	2
0	2	4	3	4	4	0	4	0	3	2
2	0	4	3	4	4	0	3	0	0	2
1	1	0	2	3	0	3	0	2	0	6
1	1	3	0	3	0	0	2	0	2	6
1	1	3	2	0	3	0	0	2	0	6
0	0	2	6	2	2	2	6	6	6	0

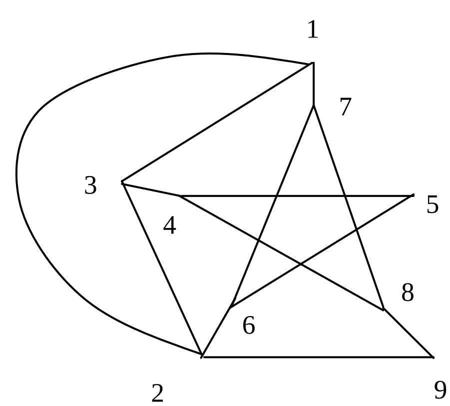


Graph G_1, n=11

Fig. 4

Preparation with *n=9* is G_0 (fig. 5) with *s*-triple *{1, 9, 5}*. Thus we get *1-H*-graph with *18* vertices (fig. 6).

1	2	3	4	5	6	7	8	9
0	1	3	1	0	1	2	1	1
1	0	1	0	1	1	0	0	3
3	1	0	2	2	0	1	1	3
1	0	2	0	3	0	0	1	1
0	1	2	3	0	3	1	1	3
1	1	0	0	3	0	2	0	2
2	0	1	0	1	2	0	2	1
1	0	1	1	1	0	2	0	3
1	3	3	1	3	2	1	3	0



Graph G_0, n=9

Fig. 5

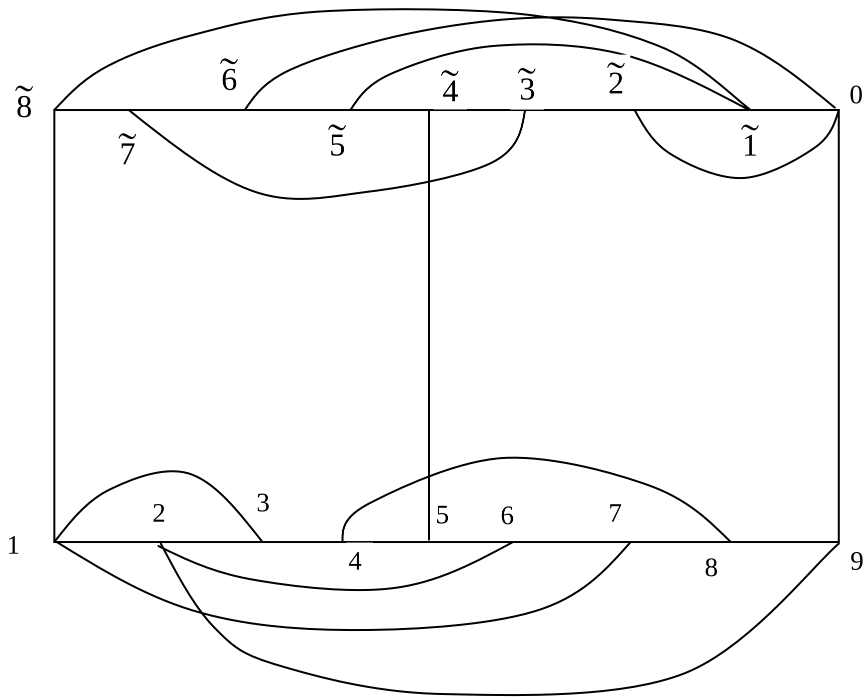


Fig. 6

Thus we get threeconnected *1-H*-graph with $n=18$ vertices. Vertices 1, 2, 0, $\tilde{1}$ are with degree four, other of degree three. It seams that at least four edge crossings. The only non-trivial automorphism is symmetry (1 0)(2 $\tilde{1}$)(3 $\tilde{2}$)(4 $\tilde{3}$)(5 $\tilde{4}$)(6 $\tilde{5}$)(7 $\tilde{6}$)(8 $\tilde{7}$)(9 $\tilde{8}$).